MATH 4030 Differential Geometry
Problem Set 3
due 20/10/2017 (Fri) at 5PM

Problems

(to be handed in)

Unless otherwise stated, we use U, O to denote connected open subsets of $\mathbb{R}^n$. The symbols S, S_1, S_2, S_3 always denote surfaces in $\mathbb{R}^3$. Recall that a *critical point* of a smooth function $f : S \to \mathbb{R}$ is a point $p \in S$ such that $df_p = 0$.

1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth functions such that $f(u) > 0$ and $g'(u) > 0$ for all $u \in \mathbb{R}$. Define a smooth map $X : \mathbb{R} \times (-\pi, \pi) \to \mathbb{R}^3$ by

$$X(u, v) := (f(u) \cos v, f(u) \sin v, g(u)).$$

Show that the image $S := X(\mathbb{R} \times (-\pi, \pi))$ is a surface, and that all the normal lines of S pass through the z-axis. What does the surface S look like?

2. Let $S \subset \mathbb{R}^3$ be a surface.

   (a) Fix a point $p_0 \in \mathbb{R}^3$ such that $p_0 \notin S$, show that the *distance function from* p_0

$$f(p) := |p - p_0|$$

   defines a smooth function $f : S \to \mathbb{R}$. Moreover, prove that $p \in S$ is a critical point of f if and only if the line joining p to p_0 is normal to S at p.

   (b) Fix a unit vector $v \in \mathbb{R}^3$, show that the *height function along* v

$$h(p) := \langle p, v \rangle$$

   defines a smooth function $h : S \to \mathbb{R}$. Prove that $p \in S$ is a critical point of h if and only if v is normal to S at p.

3. Let S be a compact connected surface, and $a \in \mathbb{R}^3$ be a unit vector.

   (a) Prove that there exists a point on S whose normal line is parallel to a.

   (b) Suppose all the normal lines of S are parallel to a. Show that S is contained in some plane orthogonal to a.

4. Let $f : S_1 \to S_2$ be a diffeomorphism between surfaces.

   (a) Show that orientability is preserved by diffeomorphisms, i.e. show that S_1 is orientable if and only if S_2 is orientable.

(b) (*Orientation-reversing diffeomorphism*) Show that the antipodal map $f : \mathbb{S}^2 \to \mathbb{S}^2$ defined by $f(x) = -x$ is an orientation-reversing diffeomorphism on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$.

5. (*Change of coordinates for tangent vectors*) Let $p \in S$ be a point on the surface S. Suppose there are two parametrizations

$$X(u,v) : U \subset \mathbb{R}^2 \to V \subset S,$$
$$\overline{X}(\overline{u},\overline{v}) : \overline{U} \subset \mathbb{R}^2 \to \overline{V} \subset S$$

such that $p \in V \cap \overline{V}$. Let $\psi = \overline{X}^{-1} \circ X : X^{-1}(V \cap \overline{V}) \to \overline{X}(V \cap \overline{V})$ be the transition map which can be written in (u,v) and $(\overline{u},\overline{v})$ coordinates as

$$\psi(u,v) = (\overline{u}(u,v), \overline{v}(u,v)).$$

If $v \in TpS$ is a tangent vector which can be expressed in local coordinates as

$$a_1 \frac{\partial X}{\partial u} + a_2 \frac{\partial X}{\partial v} = v = b_1 \frac{\partial \overline{X}}{\partial \overline{u}} + b_2 \frac{\partial \overline{X}}{\partial \overline{v}},$$

where the partial derivatives are evaluated at the point $X^{-1}(p)$ and $\overline{X}^{-1}(p)$ respectively, show that

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial \overline{u}}{\partial u} & \frac{\partial \overline{u}}{\partial v} \\ \frac{\partial \overline{v}}{\partial u} & \frac{\partial \overline{v}}{\partial v} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

where the partial derivatives are evaluated at the point $(u_0, v_0) = X^{-1}(p)$. In other words, the tangent vector transforms by multiplying the *Jacobian matrix* of the transition map ψ.

6. Find the area of the torus of revolution S defined by

$$S = \{(x,y,z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - a)^2 + z^2 = r^2\},$$

where $a > r > 0$ are given positive constants.

7. Calculate the mean curvature H and Gauss curvature K of the following surfaces:

$$S_1 = \{(x,y,z) \in \mathbb{R}^3 : z = x^2 + y^2\},$$
$$S_2 = \{(x,y,z) \in \mathbb{R}^3 : z = x^2 - y^2\},$$

with respect to the "upward" (toward positive z-axis) pointing unit normal N. Express the second fundamental form A of each surface at $p = (0,0,0)$ as a diagonal matrix. What are the principal curvatures and principal directions? Sketch the surfaces near $(0,0,0)$.

8. Show that an orientable connected surface S with $H \equiv 0 \equiv K$ must be contained in a plane.

9. Show that an *ellipsoid*

$$S = \left\{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\},$$

where $a, b, c > 0$ are constants, has positive Gauss curvature at every point.

Suggested Exercises

(no need to hand in)

1. Prove the *Chain Rule*: if $f : S_1 \to S_2$ and $g : S_2 \to S_3$ are smooth maps between surfaces, then for any $p \in S_1$,
$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

2. Let $a \in \mathbb{R}$ be a regular value of a smooth function $F : O \subset \mathbb{R}^3 \to \mathbb{R}$. Prove that the surface $S = F^{-1}(a)$ is orientable.

3. Let $f : S_1 \to S_2$ be a smooth map between surfaces where S_2 is orientable. If f is a local diffeomorphism at every $p \in S_1$, prove that S_1 is also orientable.

4. (*Invariance under rigid motions*) Let $S \subset \mathbb{R}^3$ be an orientable surface and $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion of $\mathbb{R}^3$, i.e. $\phi(p) = Ap + b$ for some $A \in O(3)$ and $b \in \mathbb{R}^3$. Let $S' = \phi(S)$ be the image surface of S under ϕ.
   (a) If $N : S \to \mathbb{S}^2$ is a Gauss map for S, prove that $N' = A(N \circ \phi^{-1}) : S' \to \mathbb{S}^2$ is a Gauss map for S'.
   (b) Let A and A' be the second fundamental for S and S' respectively (with respect to N and N' in (a)). Show that for any $p \in S$ and $v, w \in T_pS$,
$$A'_{\phi(p)}(d\phi_p(v), d\phi_p(w)) = A_p(v, w).$$
   (c) Find the relation between the mean curvature and Gauss curvature of S and S'.

5. Suppose that a surface S and a plane P are tangent along the trace of a regular curve $\alpha : I \to \mathbb{R}^3$ with $\alpha(I) \subset P$. Show that the Gauss curvature of S vanishes at every point on this curve.

6. If a surface S contains a straight line $\ell \subset S$, show that the Gauss curvature K of S satisfies $K \leq 0$ at every point on ℓ.